

On the Derivation of the L.S.Z.-Reduction Formalism in Functional Quantum Theory

H. Stumpf

Institut für Theoretische Physik, Universität Tübingen

Z. Naturforsch. **36a**, 1021–1023 (1981); received July 18, 1981

In functional quantum theory the S -matrix of a quantized relativistic field can be expressed by the scalar product of functional advanced and retarded scattering states. It is shown that for pointlike particles this scalar product can be reformulated in accordance with the results of the L.S.Z.-reduction technique for the S -matrix calculation in conventional quantum field theory. The reformulation is performed for the case of relativistic potential scattering but can easily be extended to other cases where the pointlike particles occur.

Functional quantum theory is a new formulation of quantum theory and a new field theoretic calculation method which allows the treatment of quantized fields with positive metric as well as with indefinite metric beyond perturbation theory. It was developed by Stumpf and coworkers, cf. Stumpf [1]. In particular, it can be applied to the evaluation of Heisenberg's quantized nonlinear spinor field equation with dipole ghost regularization, cf. Heisenberg [2] or to the evaluation of higher order nonlinear spinor field equations as given, for instance, by the lepton-quark model of Stumpf [3]. In these models all physical observable particles are assumed to be described by nonlinear functionals of the spinor field operators, i.e. they are assumed to be relativistic clusters resp. extended composite particles. Therefore, any comparison of results obtained from such theories with experimental facts has to be done with respect to cluster representations, i.e. the observables are to be defined with respect to cluster states. This is part of the program of functional quantum theory and the essential quantum observables are formulated in this way. In particular the S -matrix is expressed by the functional scalar product of advanced and retarded functional cluster scattering states. Then the question arises whether the equivalence of this formulation with the results of conventional quantum field theory can be established or not in cases when both methods are applicable. Concerning the S -matrix in conventional quantum field theory this

quantity can be expressed for pointlike particles via the so-called reduction formalism by means of time-ordered Greenfunctions, cf. Lehmann, Symanzik, and Zimmermann [4]. This formalism was extended to include local composite particles by Zimmermann [5]. Huang and Weldon [6] generalized the reduction formalism to extended composite particles. A criticism of the latter approach from the viewpoint of functional quantum theory will be given in subsequent papers. In this paper we restrict ourselves to the derivation of the original L.S.Z.-formalism for pointlike particles. For simplicity we treat potential scattering for one particle. The generalization to many-particle scattering will be treated elsewhere.

Theorem: Let $\{|a_i(0)\rangle^\pm = |a_i^\pm\rangle\}$ be a complete set of advanced resp. retarded Heisenberg states for one-particle potential scattering of a quantized spinorfield $\psi(x)$ with the Hamiltonian H , and let $\{|a_i(0)\rangle^f = |a_i^f\rangle\}$ be the set of corresponding incoming resp. outgoing free wave packets which are solutions of a corresponding Dirac equation with mass m . If the corresponding Schrödinger states $\{|a_i(\vartheta)\rangle^\pm\}$ satisfy the asymptotic conditions

$$\lim_{\vartheta \rightarrow +\infty} |a_i(\vartheta)\rangle^\mp = \lim_{\vartheta \rightarrow -\infty} |a_i(\vartheta)\rangle^f \quad (1)$$

for the S -matrix S_{ij} , then the following relation holds

$$\begin{aligned} S_{ij} &:= \int \sigma(x') |a_i^-\rangle^\times \varphi(x') |a_j^+\rangle d^4x' \\ &= \delta_{ij} + i \int \bar{\varphi}(x') |a_i^f\rangle \cdot \\ &\quad \cdot [(-i\gamma^\mu \delta_{\mu'} + m) \varphi(x') |a_j^+\rangle] d^4x', \quad (2) \end{aligned}$$

where $\varphi(x)$ resp. $\sigma(x)$ are the components of the corresponding functional state representations.

Reprint requests to Dr. H. Stumpf, Institut für Theoretische Physik der Universität Tübingen, Auf der Morgenstelle 14, D-7400 Tübingen.

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Proof: With respect to the functional state representation we refer to [1] and to Stumpf [7]. Due to the translational invariance we have

$$S_{ij} := -\langle a_i(0) | a_j(0) \rangle^+ \\ = \lim_{\vartheta \rightarrow \infty} -\langle a_i(\vartheta) | a_j(\vartheta) \rangle^+ \quad (3)$$

and from (1) it follows

$$S_{ij} = \lim_{\vartheta \rightarrow \infty} {}^f \langle a_i(\vartheta) | a_j(\vartheta) \rangle^+. \quad (4)$$

The functional state representations can be equally well derived for Schrödinger states at arbitrary times ϑ . This yields for (3) and (4)

$$S_{ij} = \lim_{\vartheta \rightarrow \infty} \int \sigma(x' | a_i(\vartheta)^-)^\times \varphi(x' | a_j(\vartheta)^+) d^4 x' \\ = \lim_{\vartheta \rightarrow \infty} \int \sigma(x' | a_i(\vartheta)^f)^\times \varphi(x' | a_j(\vartheta)^+) d^4 x'. \quad (5)$$

The further proof can be done in four steps.

i) We first consider the function $\sigma(x|a)$. According to [1, 7] it is defined by

$$\sigma(x|a) := {}^1 \langle x|a \rangle \quad (6)$$

with

$$\langle 0 | \psi(x) | x' \rangle^1 = \delta(x - x'). \quad (7)$$

In particular if we study $\sigma(x|a^f)$, relation (7) must be formulated for free field operators $\psi^f(x)$ and this leads to

$$\langle 0 | \psi^f(x) | x' \rangle^1 = \delta(x - x'). \quad (8)$$

Putting $y = x + \vartheta$, $y' = x' + \vartheta$, we obtain by formulating (8) with respect to y and y' the relation

$$\langle 0 | \psi^f(x + \vartheta) | x' + \vartheta \rangle^1 = \delta(x - x') \quad (9)$$

and from this it follows

$$\langle 0 | \psi^f(x) e^{-iH_0\vartheta} | x' + \vartheta \rangle^1 = \delta(x - x') \quad (10)$$

and by comparison with (8)

$$|x' + \vartheta \rangle^1 = e^{iH_0\vartheta} |x' \rangle^1. \quad (11)$$

Hence we have

$$\sigma(x|a(\vartheta)^f) = {}^1 \langle x|a(\vartheta) \rangle^f = {}^1 \langle x|e^{-iH_0\vartheta}|a(0) \rangle^f \\ = {}^1 \langle x + \vartheta|a(0) \rangle^f \\ = \sigma(x + \vartheta|a(0)^f). \quad (12)$$

ii) Next we consider $\varphi(x|a)$. It is

$$\varphi(x|a(\vartheta)^+) = \langle 0 | \psi(x) | a(\vartheta) \rangle^+ \\ = \langle 0 | \psi(x) e^{-iH\vartheta} | a(0) \rangle^+ \\ = \langle 0 | e^{iH\vartheta} \psi(x) e^{-iH\vartheta} | a(0) \rangle^+ \\ = \varphi(x + \vartheta|a(0)^+). \quad (13)$$

iii) The function $\sigma(x|a)$ is for a free field state $|a\rangle \equiv |a\rangle^f$ the contravariant representation of a Dirac one-particle state. From this it follows that we have to put

$$\sigma(x|a^f)^\times = \varphi(x|a^f)^\times \delta(t) \quad (14)$$

as by this choice the ordinary scalar product of Dirac spinors and their state representations are reproduced. Substitution of (14) into (5) yields

$$S_{ij} = \lim_{\vartheta \rightarrow \infty} \int \varphi(r', 0 | a_i(\vartheta)^f)^\times \\ \cdot \varphi(r', 0 | a_j(\vartheta)^+) d^3 r' \quad (15)$$

and with (12) and (13) we obtain

$$S_{ij} = \lim_{\vartheta \rightarrow \infty} \int \varphi(r', \vartheta | a_i(0)^f)^\times \\ \cdot \varphi(r', \vartheta | a_j(0)^+) d^3 r' \\ \equiv \lim_{\vartheta \rightarrow \infty} \int \varphi(r', \vartheta | a_i^f)^\times \varphi(r', \vartheta | a_j^+) d^3 r'. \quad (16)$$

We now apply the reduction formula

$$\lim_{\vartheta \rightarrow \infty} \int \varphi(r', \vartheta | a_i^f)^\times \varphi(r', \vartheta | a_j^+) d^3 r' \\ = \int \gamma^0 \frac{\partial}{\partial t'} [\bar{\varphi}(x' | a_i^f) \varphi(x' | a_j^+)] d^4 x' \\ + \lim_{\vartheta \rightarrow -\infty} \int \varphi(r', \vartheta | a_i^f)^\times \varphi(r', \vartheta | a_j^+) d^3 r'. \quad (17)$$

By means of

$$\lim_{\vartheta \rightarrow -\infty} \varphi(r', \vartheta | a_j^+) = \lim_{\vartheta \rightarrow -\infty} \varphi(r', 0 | a_j^+(\vartheta)) \\ = \lim_{\vartheta \rightarrow -\infty} \varphi(r', 0 | a_j^f(\vartheta)) \quad (18)$$

and of

$$\lim_{\vartheta \rightarrow -\infty} \varphi(r', \vartheta | a_i^f) = \lim_{\vartheta \rightarrow -\infty} \varphi(r', 0 | a_i^f(\vartheta)) \quad (19)$$

we have

$$\lim_{\vartheta \rightarrow -\infty} \int \varphi(r', \vartheta | a_i^f)^\times \varphi(r', \vartheta | a_j^+) d^3 r' \\ = \lim_{\vartheta \rightarrow -\infty} \int \varphi(r', 0 | a_i^f(\vartheta))^\times \varphi(r', 0 | a_j^f(\vartheta)) d^3 r' \\ = \lim_{\vartheta \rightarrow -\infty} {}^f \langle a_i(\vartheta) | a_j(\vartheta) \rangle^f \\ = {}^f \langle a_i(0) | a_j(0) \rangle^f = \delta_{ij} \quad (20)$$

and hence (16) can be written

$$S_{ij} = \delta_{ij} + \int \gamma^0 \frac{\partial}{\partial t'} [\bar{\varphi}(x' | a_i^f) \varphi(x' | a_j^+)] d^4 x'. \quad (21)$$

iv) In the last step we add the divergence

$$\int \nabla' \cdot [\bar{\varphi}(x' | a_i^f) \boldsymbol{\gamma} \varphi(x' | a_j^+)] d^4 x' \equiv 0 \quad (22)$$

which vanishes identically as we assume that the wave packets are located for $a_i^f(0)$ only in a finite part of the space. Then (21) can be written

$$S_{ij} = \delta_{ij} + \int \left(\gamma^0 \frac{\partial}{\partial t'} + \nabla' \cdot \boldsymbol{\Upsilon} \right) \cdot [\bar{\varphi}(x' | a_i^f) \varphi(x' | a_j^+)] d^4 x'. \quad (23)$$

Now $\bar{\varphi}(x' | a_i^f)$ satisfies the equation

$$\bar{\varphi}(x | a_i^f) \left(\gamma^0 \frac{\partial}{\partial t} + \nabla \cdot \boldsymbol{\Upsilon} \right) = i m \bar{\varphi}(x | a_i^f). \quad (24)$$

Using this and performing the differentiation we finally obtain

$$S_{ij} = \delta_{ij} + \int \bar{\varphi}(x' | a_i^f)^\times \cdot \left[\gamma^0 \frac{\partial}{\partial t'} + \nabla' \cdot \boldsymbol{\Upsilon} + i m \right] \varphi(x' | a_j^+) d^4 x' \quad (25)$$

which is equivalent to (2). Q.E.D.

Formulae of the type (2) are already contained in the paper of Freese [8]. The difference between this approach and Freese's consists in the fact that Freese did not use the dual functional state representations $|\mathfrak{S}(a)\rangle$ and $|\mathfrak{F}(a)\rangle$ for a state $|a\rangle$. Therefore a scalar product of the type $\langle a|b\rangle = \langle \mathfrak{S}(a) | \mathfrak{F}(b) \rangle$ was not known to him. The advantage

of our approach therefore consists in a systematic deduction from first principles while Freese practically guessed his formulae.

Finally, a comment should be given with respect to ghost states. Ghost states satisfy in their covariant representation, in the simplest case, an equation of the kind

$$(-i \gamma^\mu \partial_\mu + m)^2 \varphi(x) = 0. \quad (26)$$

Particular solutions are given for instance

$$\begin{aligned} \varphi(x|g) &= e^{ipx}, \\ \varphi(x|d) &= (A + B x^0 \partial_0) e^{ipx}, \end{aligned} \quad (27)$$

where φ_g is a ghost solution, while φ_d is a dipole ghost solution. The corresponding contravariant functions can be chosen in the following way

$$\begin{aligned} \sigma(x|g) &= e^{ip \cdot x} [a_{1g} \delta(x_0) + a_{2g} \delta'(x_0)], \\ \sigma(x|d) &= e^{ip \cdot x} [a_{1d} \delta(x_0) + a_{2d} \delta'(x_0)], \end{aligned} \quad (28)$$

where the coefficients $a_{ig}, a_{id}, i = 1, 2$ can be determined by comparison with the metrical tensor g_{ab} . Then similar reduction formulae could be derived for ghost scattering. However, confinement has to be taken into account which requires another treatment of the S -matrix. This will be discussed elsewhere.

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